Modeling Term Structures of Defaultable Bonds

Duffie and Singleton (1999)

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Outline

- Introduction of defaultable claims modeling
- Consider alternative recovery methods
- Valuation of defaultable bonds

Review

- For a contingent claim X at T, given its real-world cont'd return μ:

$$V_0 = e^{-\mu T} E[X]$$

- Using the equivalent martingale approach:

$$V_0 = e^{-rT} E^Q[X]$$

- If the risk-free rate r is random process (this is the case in most fixed-income modelling)

$$V_0 = E^Q\left[\exp\left(-\int_0^T r_t dt\right) X\right]$$

Hazard Rate

- Survival function: $S(t) = prob(T > t)$, which is decreasing
- Default probability: $S(t) - S(t + \Delta t) = prob(t \leq\ T < t + \Delta t)$
- Conditional default probability:

$$\frac{S(t) - S(t+\Delta t)}{S(t)} = \frac{prob(t \leq T < t + \Delta t)}{prob(T > t)} = prob(T < t + \Delta t \mid T > t)$$

- "Density" of conditional default probability: $\frac{S(t) - S(t+\Delta t)}{\Delta t \cdot S(t)}$
- Hazard rate: $h(t) = \lim_{\Delta t \to 0} \frac{S(t) - S(t+\Delta t)}{\Delta t \cdot S(t)}$
- As a result, the conditional default probability in a short time interval dt can be written as $h(t)dt$

Intuition

- Short rate process r_t and equivalent martingale measure Q
- Let h_t denotes the hazard rate for default at time t
- Let L_t denotes the expected fractional loss in market value if default were to occur at time t, conditional on $\mathcal{F}_t$
- The initial market value of the defaultable claim to X is

$$V_0 = E^Q\left[\exp\left(-\int_0^T R_t dt\right)X\right]$$

  where the default-adjusted short-rate process $R_t = r_t + h_t L_t$
- Need to be proven under both discrete and continuous settings

Defaultable Claims in Discrete Space

- Let φ_s denotes the dollar amount of recovery given default at time s. What's the market value of an asset V_t, given future recovery φ_{t+1} given default and future value V_{t+1} given no default?

$$V_t = h_t e^{-r_t} E_t^Q[\varphi_{t+1}] + (1 - h_t) E_t^Q[V_{t+1}]$$

- Recursively solving forward...

$$V_t = E_t^Q\left[\sum_{j=0}^{T-1} h_{t+j}\, e^{-\sum_{k=0}^{j} r_{t+k}} \varphi_{t+j+1} \prod_{l=0}^{j} (1 - h_{t+l-1})\right] +$$

$$E_t^Q\left[e^{-\sum_{k=0}^{T-1} r_{t+k}} \varphi_{t+T} \prod_{j=1}^{T} \left(1 - h_{t+j-1}\right)\right]$$

Defaultable Claims in Discrete Space

- Suppose we adapt "RMV" (recovery of market value) assumption here, i.e., take the RN expected recovery as a fraction of RN expected survival contingent market value.

$$E_s^Q[\varphi_{s+1}] = (1 - L_s)E_s^Q[V_{s+1}]$$

- Substitute it into the V_t expression:

$$V_t = h_t e^{-r_t}(1 - L_t)E_t^Q[V_{t+1}] + (1 - h_t)e^{-r_t}E_t^Q[V_{t+1}]$$

$$= E_t^Q[e^{-\sum_{j=0}^{T-1} R_{t+j}}X_{t+T}]$$

- Where $e^{-R_t} = (1 - h_t)e^{-r_t} + h_t e^{-r_t}(1 - L_t)$

- Or $R_t = r_t + h_t L_t$

Defaultable Claims in Discrete Space

- Why this representation is good?
- If we assume that h_t and L_t are exogenous process, we can just model R for the defaultable bonds, instead of r, using single- or multifactor model such as CIR or Vasicek, or HJM model.
- State Dependence is accommodated, i.e., h_t and L_t may be correlated with each other, with r_t, with economic cycle...
- If the exogeneity is violated, we must find other methods. (For example, market value of recovery is fixed..)

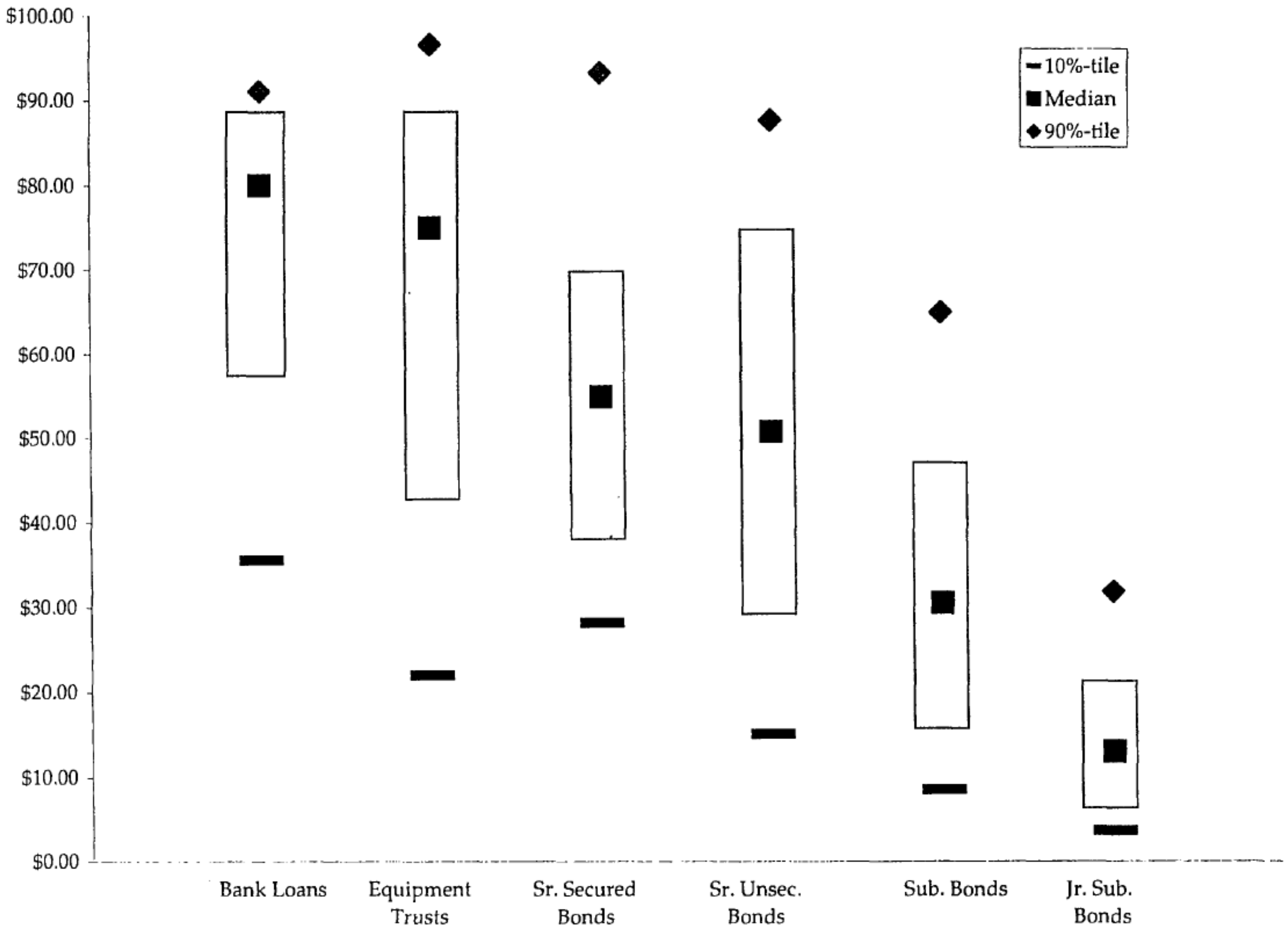

Figure 1
Distributions of recovery by seniority

Defaultable Claims in Continuous Space

- Contingent claim (Z, τ): random variable Z and stopping time τ where Z is paid. Z is $\mathcal{F}_\tau$ measurable.
- The ex-dividend price process U for (Z, τ) is given by:

$$U_t = E_t^Q \left[\exp\left(-\int_t^\tau r_u du \right) Z \right]$$

- Defaultable claim $((X, T), (X', T'))$: (X, T) is the obligation of issuer to pay X at T. (X', T') defines the stopping time T' when the issuer defaults and X' is recovered.
- Actual claim (Z, τ) generated by such a defaultable claim is defined by:

$$\tau = \min(T, T') \qquad Z = X 1_{\{T < T'\}} + X' 1_{\{T \geq T'\}}$$

Defaultable Claims in Continuous Space

- Note that T' is random by nature since we don't know when the issuer defaults.
- We model T' by setting a variable $\Lambda_t = 1_{\{t \geq T'\}}$
- From the definition of hazard rate, we know that the instantaneous conditional default probability can be written as $h_t dt$. However, in this case, a defaultable claim can only default once. Once it defaults the probability will become one and will never change. To model this, we rewrite the probability as $(1 - \Lambda_t)h_t dt$
- After adding a demean process M_t, we can get

$$d\Lambda_t = (1 - \Lambda_t)h_t dt + dM_t$$

Defaultable Claims in Continuous Space

- The payoff X' at default is also random. It is modeled as

$$X' = (1 - L_t)U_{t-}$$

- where $U_{t-} = \lim_{s \uparrow t} U_s$ is the price of the claim ”just before” default
- Key assumption is that this L_t is predictable by the information up to t, i.e., $\mathcal{F}_t$

Defaultable Claims in Continuous Space

- We know that if we discounted the gain from an asset by the short-rate process r, the gain process must be a martingale under Q.
- The discounted gain process G is defined by:

$$G_t = \exp\left(-\int_0^t r_s ds\right) V_t (1 - \Lambda_t) + \int_0^t \exp\left(-\int_0^s r_u du\right)(1 - L_s) V_{s-} d\Lambda_s$$

- This is a martingale. What should we do to get V_t? Ito's Lemma.
- Let $dG_t = 0$ and use the fact that $X' = (1 - L_t) U_{t-}$, we can get

$$V_t = \int_0^t R_s V_s ds + m_t$$

Defaultable Claims in Continuous Space

- Given the terminal boundary condition $V_T = X$, we can get:

$$V_0 = E^Q \left[\exp\left(-\int_0^T R_t dt \right) X \right]$$

where $R_t = r_t + h_t L_t$

- We can see another advantage of this model. In its final form, we can get rid of (X', T'), Λ_t and U_t. That being said, we don't need to model the characteristics of the defaultable claim. Instead, only by considering the non-defaultable contingent claim and changing the discount rate can get the final answer.

Some Special Cases

- Continuous-time Markov formulation: Assume a state variable process Y that is Markovian

$$J(Y_t, t) = E^Q\left[\exp\left(-\int_t^T \rho(Y_s)ds\right) g(Y_T) \mid Y_t\right]$$

- $Y_t = (Y_{1t}, Y_{2t}, \dots, Y_{nt})'$ solves a SDE:

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dB_t$$

- J solves the backward Kolmogorov PDE:

$$J_t(y,t) + J_y(y,t)\mu(y) + \frac{1}{2} trace\left(J_{yy}(y,t)\sigma(y)\sigma'(y)\right) - \rho(y)J(y,t) = 0$$

with boundary condition

$$J(y,T) = g(y)$$

Some Special Cases

- Price-dependent expected loss rate:

$$J(Y_t, t) = E^Q\left[\exp\left(-\int_t^T \rho(Y_s, J(Y_s, s))ds\right) g(Y_T) \mid Y_t\right]$$

  Corresponding PDE can be treated numerically
- Uncertainty about recovery:

$$X' = (1 - l)U_{T'-}$$

  where l is a bounded, $\mathcal{F}_{T'}$ measurable random variable
- L_t is the expectation of l given all info up to but not including time t.
- $L_{T'} = E(l|\mathcal{F}_{T'-})$
- With this change, the pricing formula $R_t = r_t + h_t L_t$ still applys.

Defaultable Bonds: Recovery and valuation

- Consider the following recovery methods:

RT: $\varphi_t = (1 - L_t)P_t$, where L is an exogenously specified fractional recovery process and P_t is the price at time t of an otherwise equivalent, default-free bond [Jarrow and Turnbull (1995)].

RFV: $\varphi_t = (1 - L_t)$; the creditor receives a (possibly random) fraction $(1 - L_t)$ of face (\$1) value immediately upon default [Brennan and Schwartz (1980) and Duffee (1998)].

- Under RT, the computational burden of directly computing V_t can be substantial. Time of default, the joint $\mathcal{F}_t$-conditional distributions of L_v, h_s, r_u for all v, s, u between t and T plays a computationally challenging role in determining V_t.

Defaultable Bonds: Recovery and valuation

- RMV: $E_s^Q[\varphi_{s+1}] = (1 - L_s)E_s^Q[V_{s+1}]$
- RMV vs RFV: RMV matched to the legal structure of swap contract better. RMV model is more convenient for corporate bonds because we can just apply standard default-free term-structure modelling techniques. RFV, on the other hands, is more realistic when absolute priority applies.
- Is there a significant difference between RMV and RFV model?
- For simplicity, we take $L_t = \bar{L}$, a constant. We model r and h by

$$r_t = \rho_0 + Y_t^1 + Y_t^2 - Y_t^0$$
$$h_t = bY_t^0 + Y_t^3$$

where Y_t^i is “square root diffusions” under Q

Defaultable Bonds: Recovery and valuation

- Under RMV assumption:

$$V_{nt}^{RMV} = cE_t^Q\left(\sum_{j=1}^{2n} e^{-\int_t^{t+.5j} R_s\,ds}\right) + E_t^Q\left(e^{-\int_t^{t+n} R_s\,ds}\right)$$

where $R_t = r_t + h_t\bar{L}$

- Under RFV assumption:

$$V_{nt}^{RFV} = cE_t^Q\left(\sum_{j=1}^{2n} e^{-\int_t^{t+0.5j}(r_s+h_s)\,ds}\right) + E_t^Q\left(e^{-\int_t^{t+n}(r_s+h_s)\,ds}\right) + \int_t^{t+n}(1-\bar{L})\gamma(Y_t,t,s)\,ds,$$

where $\gamma(Y_t,t,s) = E_t^Q\left(h_s e^{-\int_t^s (r_u+h_u)\,du}\right)$

Defaultable Bonds: Recovery and valuation

- Calibrate the RMV and RFV model:
- Bonds with fixed ten-year par-coupon spreads. (known c)
- Fixed $L_t = \bar{L}$
- r_t and h_t are modelled by several square-root diffusion processes
- Minimizing the error between model estimated bond prices and real bond prices through changing the parameters of r_t and h_t.
- Compute the mean implied intensity $\bar{h}$

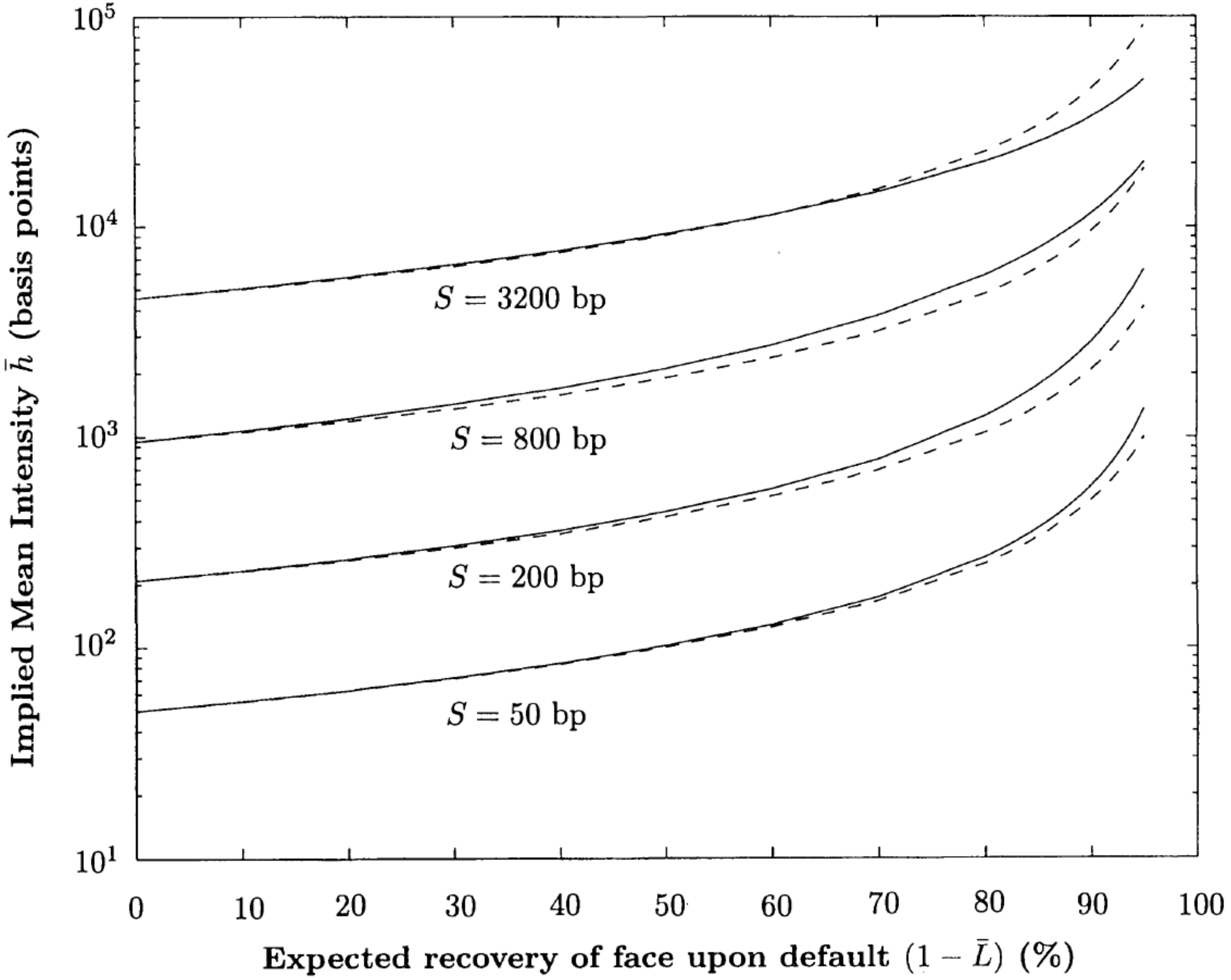

Figure 2
For fixed ten-year par-coupon spreads, S, this figure shows the dependence of the mean hazard rate $\bar{h}$ on the assumed fractional recovery $1-\overline{L}$. The solid lines correspond to the model *RFV*, and the dashed lines correspond to the model *RMV*.

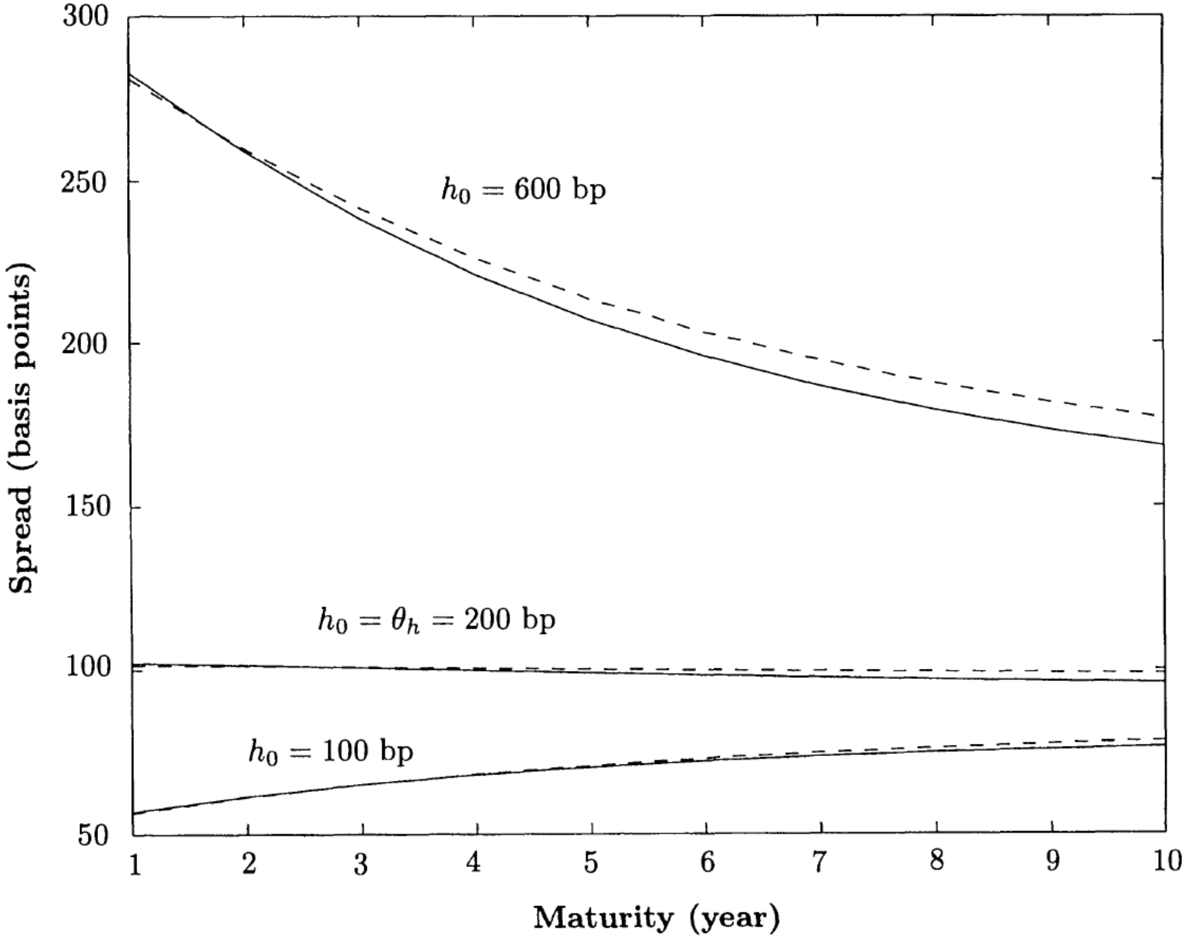

Figure 3
Term structures of par-coupon yield spreads for *RMV* (dashed lines) and *RFV* (solid lines), with 50% recovery upon default, a long-run mean hazard rate of $\theta_h = 200$ bp, a mean reversion rate of $\kappa = 0.25$, and an initial hazard-rate volatility of 100%.

Noncallable Corporate Bonds

- Note that the hazard rate process h_t and the fractional loss L_t enter the discount rate in the product from $h_t L_t$
- Knowledge of defaultable bond prices before default alone is not sufficient to separately identify h_t and L_t
- If one has prices of undefaulted junior and senior bonds of the same issuer, along with the prices of the Treasury bonds, we can extract $h_t L_t^J$ and $h_t L_t^S$, thus can infer L_t^J / L_t^S.
- We can just model jointly the dynamic properties of r_t and the "short spread" $s_t \equiv h_t L_t$

Noncallable Corporate Bonds

- Case 1: Square root diffusion model of Y

$$r_t = \delta_0 + \delta_1 Y_{1t} + \delta_2 Y_{2t} + \delta_3 Y_{3t}$$

$$s_t = \gamma_0 + \gamma_1 Y_{1t} + \gamma_2 Y_{2t} + \gamma_3 Y_{3t}$$

- Dai and Singleton (1998) proposes the “most flexible” affine term structure model

$$dY_t = \mathcal{K}(\Theta - Y_t)dt + \sqrt{S_t}dB_t$$

where $\mathcal{K}$ is a 3*3 matrix with positive diagonal and nonpositive off-diagonal elements; Θ in $\mathbb{R}_+^3$; S_t is 3*3 diagonal matrix with diagonal elements Y_{1t}, Y_{2t} and Y_{3t}

Noncallable Corporate Bonds

- Duffie (1999) considered the special case in which $\delta_0 = -1$ and $\delta_3 = 0$, so r_t could take on negative values and depend only on the first two state variables.
- He also assumed that $\mathcal{K}$ is diagonal (Y_{1t}, Y_{2t} and Y_{3t} are independent)
- However, the only means of introducing negative correlation among r_t and s_t is to allow for negative γ, which means the hazard rate may take on negative values.
- Within this correlated square-root model of (r_t, s_t), one cannot simultaneously have a nonnegative hazard rate process and negatively correlated r_t and h_t without having one or more γ or δ negative.

Noncallable Corporate Bonds

- Case 2: More flexible correlation structure for (r_t, s_t)

$$r_t = \delta_0 + \delta_1 Y_{1t} + Y_{2t} + Y_{3t}$$

$$s_t = \gamma_0 + \gamma_1 Y_{1t} + \gamma_2 Y_{2t}$$

- We assume that

$$dY_t = \mathcal{K}(\Theta - Y_t)dt + \Sigma\sqrt{S_t}dB_t$$

where $\mathcal{K}$ is a 3*3 matrix with positive diagonal and nonpositive off-diagonal elements; Θ in $\mathbb{R}_+^3$; and

$$S_{11}(t) = Y_1(t),$$

$$S_{22}(t) = [\beta_2]_2 Y_2(t),$$

$$S_{33}(t) = \alpha_3 + [\beta_3]_1 Y_1(t) + [\beta_3]_2 Y_2(t)$$

Noncallable Corporate Bonds

- All of $\delta_0, \delta_1, \gamma_0, \gamma_1, \gamma_2$ are strictly positive
- Dai and Singleton show that in this case the most flexible and admissible affine term structure has:

$$\mathcal{K} = \begin{bmatrix} \kappa_{11} & \kappa_{12} & 0 \\ \kappa_{21} & \kappa_{22} & 0 \\ 0 & 0 & \kappa_{33}, \end{bmatrix} \quad \Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \sigma_{31} & \sigma_{32} & 1, \end{bmatrix}.$$

- The short-spread rate s_t is strictly positive. At the same time, the signs of σ_{31} and σ_{32} are unconstrained, so the third state variable may have increments that are negatively correlated with the first two.

Valuation of defaultable callable bonds

- During the time window of callability, we have the recursive pricing formula

$$V_t = \min[\overline{V}_t, e^{-R_t} E_t^Q (V_{t+1} + d_{t+1})]$$

- Outside the callability window,

$$V_t = e^{-R_t} E_t^Q (V_{t+1} + d_{t+1})$$

Valuation of defaultable callable bonds

- In a more continuous time context, let $\mathcal{T}(t, T)$ denote the set of feasible call policies. The market price at t is:

$$V_t = \min_{\tau \in \mathcal{T}(t,T)} E_t^Q \left[\sum_{t < T(i) \leq \tau} \gamma_{t,T(i)} c_i + \gamma_{t,\tau} \right]$$

where

$$\gamma_{t,s} = \exp\left(- \int_t^s R_u \, du \right)$$

- This equation can be solved by a discrete algorithm similar to the equations in the previous slide.

More (in the paper)

- Defaultable HJM model
- Pricing Credit Derivatives
- Etc.

Take home message

- The initial market value of the defaultable claim to X is
$$V_0 = E^Q\left[\exp\left(-\int_0^T R_t dt\right)X\right]$$
where the default-adjusted short-rate process $R_t = r_t + h_t L_t$
- All financial products with defaultable nature can be modeled in this way.
- If we assume that h_t and L_t are exogenous process, we can just model the process R for the defaultable bonds instead of r.

My remarks (pros)

- Very detailed.
- A variety of models regards to defaultable claims/bonds under different assumptions are given.
- These models can be directly applied in the market using market data (calibration).
- Some brief comparisons of models are given.

My remarks (cons)

- Too much theoretical stuffs, should give more calibration (I mean to be theoretical is good, but it is always better to give some empirical results)
- More like a encyclopedia instead of a paper. (always introduce a model and say “please refer to some other papers”. I think a paper should focus one or a few models and dig deeper).
- Without a clear conclusion. (which model is good or bad under which conditions?)

Thank you

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